

PARAMETER ESTIMATION UNDER GAUSSIAN MODEL UNCERTAINTIES BY ITERATIVE COVARIANCE APPROXIMATION

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ABSTRACT

We propose a novel iterative algorithm for estimating a deterministic but unknown parameter vector in the presence of Gaussian model uncertainties. This iterative algorithm is based on a system model where an overall noise term describes both, the measurement noise and the noise resulting from the model uncertainties. This overall noise term is a function of the true parameter vector. The proposed iterative algorithm can be applied on structured as well as unstructured models and it outperforms prior art algorithms for a broad range of applications.

Index Terms— total least squares, structured total least squares, model error, deconvolution, classical estimation.

1. INTRODUCTION

The Gaussian linear model

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \quad (1)$$

is frequently used in many areas of signal processing. Here, $\mathbf{y} \in \mathbb{R}^{N_y \times 1}$ is the vector of measurements, $\mathbf{x} \in \mathbb{R}^{N_x \times 1}$ is a deterministic but unknown parameter vector, $\mathbf{H} \in \mathbb{R}^{N_y \times N_x}$ is the measurement matrix and $\mathbf{n} \in \mathbb{R}^{N_y \times 1}$ is zero mean Gaussian measurement noise with known covariance matrix \mathbf{C}_{nn} . Linear classical estimators such as the least squares (LS) estimator or the best linear unbiased estimator (BLUE) [1, 2] assume that the measurement matrix \mathbf{H} is perfectly known. In practice, this assumption frequently not hold. A prominent case is where \mathbf{H} is a convolution matrix that is itself estimated from an imperfectly measured system output. The error in \mathbf{H} is often neglected since it is unknown.

In contrast to the LS estimator and the BLUE, total least squares (TLS) estimation techniques incorporate model errors. E.g., for independent and identically distributed (i.i.d.) model errors with Gaussian probability density function (PDF), the maximum likelihood (ML) solution of the TLS problem was analyzed in [3]. However, in many practical applications \mathbf{H} has some sort of structure as it is the case for Toeplitz or Hankel matrices. If this is the case, the model

errors are clearly not i.i.d. any more. Structured total least squares (STLS) techniques have been developed to deal with these kind of problems [4–6]. An overview of different TLS and STLS methods can be found in [7–9].

In this work we compare our novel approach with two iterative algorithms, which serve as performance reference in the remainder of this paper. The first one, introduced in [3], is an approach for solving the maximum likelihood (ML) problem based on classical expectation-maximization (EM) [10]. This algorithm, referred to as ML-EM algorithm, treats the model errors as random and allows for an incorporation of the model error variance. By doing so, a uniform variance for every element in \mathbf{H} was assumed. The second one represents an algorithm from the class of STLS approaches and is introduced in [11]. This iterative algorithm is called the structured total least norm (STLN) algorithm and it is capable of dealing with structured measurement matrices. This approach treats the model errors as deterministic but unknown. Hence, it prevents the usage of model error variances.

In this paper, we propose a novel iterative algorithm that combines information about the structure as well as the model error variances. Moreover, this algorithm can be employed on structured as well as unstructured problems. In contrast to the ML-EM algorithm, our algorithm is capable of incorporating different variances for every element of \mathbf{H} . A difference to the STLN algorithm is that the proposed algorithm treats the model errors as random variables, allowing to incorporate the model error variances. All three algorithms require solving an inverse linear problem at each iteration. Simulation examples are presented which show that the proposed algorithm is able to outperform both competing algorithms in a mean square error (MSE) sense for a broad range of model error and noise variances.

This iterative algorithm is based on a system model investigated in this work where an overall noise term describes both, the measurement noise and the noise resulting from the model uncertainties. The covariance matrix of this overall noise term is evaluated for different cases. Considering the model errors as random with known second order statistics is motivated by practical examples such as multiple-input

multiple-output (MIMO) communication channels or beamforming [12–15].

The remainder of this paper is organized as follows: In Sec. 2, the underlying system model is introduced. Here we distinguish between unstructured and structured measurement matrices. For the structured case, we constrain ourselves to convolution matrices. However, extensions to other kind of structured matrices are easily possible. The proposed iterative algorithm is discussed in Sec. 3. Simulation results demonstrating its performance are given in Sec. 4.

Notation:

Lower-case bold face variables ($\mathbf{a}, \mathbf{b}, \dots$) indicate vectors, and upper-case bold face variables ($\mathbf{A}, \mathbf{B}, \dots$) indicate matrices. We further use \mathbb{R} and \mathbb{C} to denote the set of real and complex numbers, respectively, $(\cdot)^T$ to denote transposition and $(\cdot)^H$ to denote conjugate transposition, $\mathbf{I}^{n \times n}$ to denote the identity matrix of size $n \times n$, and $\mathbf{0}^{m \times n}$ to denote the all-zero matrix of size $m \times n$. If the dimensions are clear from the context we simply write \mathbf{I} and $\mathbf{0}$, respectively. $E[\cdot]$ denotes the expectation operator, $[\cdot]_i$ the i^{th} element of a vector and $[\cdot]_{i,j}$ the element of a matrix at the i^{th} row and the j^{th} column.

2. SYSTEM MODEL

This section describes the underlying model used in the remainder of this paper. In a first step, the measurement matrix is assumed to be unstructured and the model uncertainties are assumed to be i.i.d. Gaussian. Afterwards, \mathbf{H} is assumed to be a structured convolution matrix built from an estimated or measured impulse response. Hence, \mathbf{H} is a special form of a Toeplitz matrix and, as it will be shown, allows for correlated model uncertainties.

2.1. Unstructured Measurement Matrices

We denote $\hat{\mathbf{H}}$ as the measured or estimated measurement matrix and assume it comes along with error variances for every entry. The error variances assembled in a matrix of the same size as $\hat{\mathbf{H}}$ is denoted as $\mathbf{V} \in \mathbb{R}^{N_y \times N_x}$. Furthermore, the errors are assumed to be i.i.d. zero mean Gaussian random variables. The measurements are modeled as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} = (\hat{\mathbf{H}} + \mathbf{B})\mathbf{x} + \mathbf{n}, \quad (2)$$

where $\mathbf{H} = \hat{\mathbf{H}} + \mathbf{B}$, with $\hat{\mathbf{H}}$ being the estimated measurement matrix and \mathbf{B} being a zero mean random matrix. In (2), \mathbf{H} and \mathbf{B} are unknown while $\hat{\mathbf{H}}$ is known. We further rewrite (2) according to

$$\mathbf{y} = \hat{\mathbf{H}}\mathbf{x} + \underbrace{\mathbf{B}\mathbf{x} + \mathbf{n}}_{\mathbf{w}} \quad (3)$$

$$= \hat{\mathbf{H}}\mathbf{x} + \mathbf{w}, \quad (4)$$

with the new overall noise vector \mathbf{w} . This noise vector combines the measurement noise with the noise from the model uncertainties. The PDF of \mathbf{w} is Gaussian since it consists of the product of a matrix with zero mean Gaussian random elements with an unknown but deterministic vector plus the Gaussian vector \mathbf{n} . Consequently, \mathbf{w} has zero mean and its covariance matrix in dependence of the unknown parameter vector \mathbf{x} can be derived as follows. Let \mathbf{b}_i^T be the i^{th} row of \mathbf{B} , then the i^{th} element of \mathbf{w} is given by

$$[\mathbf{w}]_i = \mathbf{b}_i^T \mathbf{x} + [\mathbf{n}]_i \quad (5)$$

and has the variance

$$\sigma_i^2 = [\mathbf{V}]_{i,1} |[\mathbf{x}]_1|^2 + [\mathbf{V}]_{i,2} |[\mathbf{x}]_2|^2 + \dots + [\mathbf{V}]_{N_x} |[\mathbf{x}]_{N_x}|^2 + [\mathbf{C}_{nn}]_{i,i}. \quad (6)$$

All variances assembled in a covariance matrix reads as

$$\mathbf{C}_{\mathbf{w}\mathbf{w}} = \text{diag}(\mathbf{V}|\mathbf{x}|^2) + \mathbf{C}_{nn}, \quad (7)$$

where the term $|\mathbf{x}|^2$ means a column vector of the element-wise absolute squares of the vector \mathbf{x} .

2.2. Convolution Matrices

We will now assume that \mathbf{H} is a linear convolution matrix constructed from the impulse response $\mathbf{h} \in \mathbb{R}^{N_h \times 1}$ of a linear system such that $\mathbf{H}\mathbf{x}$ describes the convolution of \mathbf{h} and \mathbf{x} . An extension to other structured measurement matrices is easily possible. Let $\mathbf{H} = \hat{\mathbf{H}} + \mathbf{B}$ have the dimension $N_y \times N_x$ where $N_y = N_x + N_h - 1$. The i^{th} column of the convolution matrices is defined as

$$\begin{aligned} [\mathbf{H}]_{:,i} &= \begin{bmatrix} \mathbf{0}^{i-1 \times 1} \\ \mathbf{h} \\ \mathbf{0}^{N_x-i \times 1} \end{bmatrix}, \quad [\hat{\mathbf{H}}]_{:,i} = \begin{bmatrix} \mathbf{0}^{i-1 \times 1} \\ \hat{\mathbf{h}} \\ \mathbf{0}^{N_x-i \times 1} \end{bmatrix}, \\ [\mathbf{B}]_{:,i} &= \begin{bmatrix} \mathbf{0}^{i-1 \times 1} \\ \mathbf{e} \\ \mathbf{0}^{N_x-i \times 1} \end{bmatrix} \quad \forall i = 1, \dots, N_x \end{aligned} \quad (8)$$

where $\hat{\mathbf{h}}$ is the estimated impulse response and \mathbf{e} is the unknown error of $\hat{\mathbf{h}}$ with known error covariance matrix $\mathbf{C}_{ee} \in \mathbb{R}^{N_h \times N_h}$. In this case, the model uncertainties of $\hat{\mathbf{H}}$ are clearly not i.i.d. any more, leading to a different calculation of $\mathbf{C}_{\mathbf{w}\mathbf{w}}$.

Let $\mathbf{b}'_i = [\mathbf{B}]_{:,i}$ denote the i^{th} column of \mathbf{B} . The subsequent column \mathbf{b}'_{i+1} can be derived by

$$\mathbf{b}'_{i+1} = \begin{bmatrix} \mathbf{0}^{1 \times N_y-1} & 0 \\ \mathbf{I}^{N_y-1 \times N_y-1} & \mathbf{0}^{N_y-1 \times 1} \end{bmatrix} \mathbf{b}'_i = \mathbf{D}\mathbf{b}'_i, \quad (9)$$

i.e. shifting down the elements of \mathbf{b}'_i by one position. With that, the product $\mathbf{B}\mathbf{x}$ in (3) follows to

$$\mathbf{B}\mathbf{x} = \mathbf{b}'_1[\mathbf{x}]_1 + \mathbf{b}'_2[\mathbf{x}]_2 + \dots + \mathbf{b}'_{N_x}[\mathbf{x}]_{N_x} \quad (10)$$

$$= [\hat{\mathbf{x}}]_1 \mathbf{b}'_1 + [\hat{\mathbf{x}}]_2 \mathbf{D}\mathbf{b}'_1 + \dots + [\hat{\mathbf{x}}]_{N_x} \mathbf{D}^{N_x-1} \mathbf{b}'_1 \quad (11)$$

$$= ([\mathbf{x}]_1 \mathbf{I} + [\mathbf{x}]_2 \mathbf{D} + \dots + [\mathbf{x}]_{N_x} \mathbf{D}^{N_x-1}) \mathbf{b}'_1 \quad (12)$$

$$= \mathbf{P}(\mathbf{x}) \mathbf{b}'_1. \quad (13)$$

With this result, \mathbf{w} follows as

$$\mathbf{w} = \mathbf{P}(\mathbf{x})\mathbf{b}'_1 + \mathbf{n} \quad (14)$$

with its covariance matrix

$$\mathbf{C}_{\mathbf{ww}} = E \left[(\mathbf{P}(\mathbf{x})\mathbf{b}'_1) (\mathbf{P}(\mathbf{x})\mathbf{b}'_1)^H \right] + \mathbf{C}_{\mathbf{nn}} \quad (15)$$

$$= \mathbf{P}(\mathbf{x})\mathbf{C}_{\mathbf{b}'_1\mathbf{b}'_1}\mathbf{P}(\mathbf{x})^H + \mathbf{C}_{\mathbf{nn}}. \quad (16)$$

The covariance matrix $\mathbf{C}_{\mathbf{b}'_1\mathbf{b}'_1}$ follows from (8) and the covariance matrix of the estimation error \mathbf{e} according to

$$\mathbf{C}_{\mathbf{b}'_1\mathbf{b}'_1} = \begin{bmatrix} \mathbf{C}_{\mathbf{ee}} & \mathbf{0}_{N_h \times N_x - 1} \\ \mathbf{0}_{N_x - 1 \times N_h} & \mathbf{0}_{N_x - 1 \times N_x - 1} \end{bmatrix} \in \mathbb{R}^{N_y \times N_y}. \quad (17)$$

Note that the PDF of \mathbf{w} is again Gaussian for the same reasons as stated in Sec. 2.1. Also note that this formulation allows for two sources of correlated model errors. The first source of correlation is the structure in $\hat{\mathbf{H}}$. The second source of correlation is $\mathbf{C}_{\mathbf{ee}}$, which describes the errors in $\hat{\mathbf{h}}$. Hence, the iterative algorithm introduced in the next section is capable of dealing with both kind of correlations.

3. ITERATIVE ALGORITHM

The BLUE applied on the linear model in (4) incorporating the true covariance matrix $\mathbf{C}_{\mathbf{ww}}$ follows as

$$\hat{\mathbf{x}}_{\text{ideal}} = \left(\hat{\mathbf{H}}^H \mathbf{C}_{\mathbf{ww}}^{-1} \hat{\mathbf{H}} \right)^{-1} \hat{\mathbf{H}}^H \mathbf{C}_{\mathbf{ww}}^{-1} \mathbf{y} \quad (18)$$

and it is referred to as *ideal* BLUE [16]. The true $\mathbf{C}_{\mathbf{ww}}$ according to (7) or (16), however, requires the knowledge of the true parameter vector. To overcome this problem, we propose the iterative algorithm described below. Its basic idea is to make an initial guess of the impulse response $\hat{\mathbf{x}}_0$ (the index denotes the algorithm's iteration number). This first guess could e.g., origin from an LS estimation which does not incorporate any noise statistics. $\hat{\mathbf{x}}_0$ is then used to estimate $\hat{\mathbf{C}}_{\mathbf{ww},0}$ in (7) or (16). This estimated covariance matrix is then incorporated by the BLUE in order to yield a better estimate $\hat{\mathbf{x}}_1$ and so on. This procedure is summarized as follows

Initialization: LS estimation

$$\hat{\mathbf{x}}_0 = \left(\hat{\mathbf{H}}^H \hat{\mathbf{H}} \right)^{-1} \hat{\mathbf{H}}^H \mathbf{y};$$

for $k \leftarrow 0$ **to** N_{iter} **do**

 estimate $\mathbf{C}_{\mathbf{ww},k}$ according to (7) or (16) using $\hat{\mathbf{x}}_k$
 instead of \mathbf{x} ;

$$\hat{\mathbf{x}}_{k+1} = \left(\hat{\mathbf{H}}^H \hat{\mathbf{C}}_{\mathbf{ww},k}^{-1} \hat{\mathbf{H}} \right)^{-1} \hat{\mathbf{H}}^H \hat{\mathbf{C}}_{\mathbf{ww},k}^{-1} \mathbf{y};$$

end

Algorithm 1: proposed algorithm

The proposed algorithm is of similar complexity as the ML-EM and STLN algorithms. It performs a weighting of

the measurements according to $\hat{\mathbf{C}}_{\mathbf{ww},k}$, which incorporates the model error variances as well as the measurement noise variances. In the case of $\hat{\mathbf{H}}$ being a convolution matrix, even the covariances of the estimated impulse response are considered in order to improve the estimation.

The proposed algorithm is unbiased when averaged over the PDF of \mathbf{n} and \mathbf{B} , and biased when only averaged over the PDF of \mathbf{n} . Let $\mathbf{E}_k = \left(\hat{\mathbf{H}}^H \hat{\mathbf{C}}_{\mathbf{ww},k}^{-1} \hat{\mathbf{H}} \right)^{-1} \hat{\mathbf{H}}^H \hat{\mathbf{C}}_{\mathbf{ww},k}^{-1}$ denote the estimator matrix at iteration k , then it holds that $\mathbf{E}_k \hat{\mathbf{H}} = \mathbf{I}$, independent of the estimated parameter vector at the previous iteration cycles. Consequently, the conditional expected vector of $\hat{\mathbf{x}}_{k+1}$ for fixed \mathbf{B} follows to

$$\begin{aligned} E_{\mathbf{n}}[\hat{\mathbf{x}}_{k+1}|\mathbf{B}] &= E_{\mathbf{n}}[\mathbf{E}_k \mathbf{y}|\mathbf{B}] \\ &= E_{\mathbf{n}}[\mathbf{E}_k \hat{\mathbf{H}} \mathbf{x} + \mathbf{E}_k \mathbf{B} \mathbf{x} + \mathbf{E}_k \mathbf{n}|\mathbf{B}] \\ &= \mathbf{x} + \mathbf{E}_k \mathbf{B} \mathbf{x}. \end{aligned} \quad (19)$$

Since $E_{\mathbf{B}}[\mathbf{B}] = \mathbf{0}$, unbiasedness is ensured when averaged over the PDF of \mathbf{n} and \mathbf{B} .

Simulations showed that the main performance gain is usually achieved after the first iteration. However, there exists at least one case where the iterations yield no performance gain. If $\hat{\mathbf{C}}_{\mathbf{ww},k}$ is a scaled identity matrix, the proposed algorithm reduces to the ordinary LS estimator, preventing any performance increase. This is, e.g., the case when the following two conditions hold: a) The measurement matrix is unstructured and \mathbf{V} has the same variance at every element. b) the noise covariance matrix $\mathbf{C}_{\mathbf{nn}}$ is a scaled identity matrix.

We note that a similar iterative application of the BLUE was applied in [16–18] for channel impulse response estimation in wireless communication applications. Compared to them, the proposed algorithm is applicable to various applications with structured or unstructured model uncertainties. In [19] investigations of a similar procedure as the presented algorithm can be found for a very simplified model compared to the investigations in this work.

4. SIMULATION RESULTS

In this example, $\mathbf{H} \in \mathbb{R}^{7 \times 3}$ is a convolution matrix and describes the discrete convolution of the impulse response $\mathbf{h} \in \mathbb{R}^{5 \times 1}$ with $\mathbf{x} \in \mathbb{R}^{3 \times 1}$. For the simulations, the impulse responses were randomly generated with mean $E[\mathbf{h}] = \mathbf{0}^{5 \times 1}$ and covariance matrix $\mathbf{C}_{\mathbf{hh}} = \mathbf{I}^{5 \times 5}$. The parameter vector was chosen to be $\mathbf{x} = [1 \quad 0.5 \quad 0.25]^T$.

For the first analysis, the noise covariance matrix was a scaled identity matrix $\mathbf{C}_{\mathbf{nn}} = \sigma_{\mathbf{n}}^2 \mathbf{I}^{7 \times 7}$, where the scaling factor $\sigma_{\mathbf{n}}^2$ was varied between 10^{-8} and 10^{-3} . The impulse response estimation step was assumed to yield the error covariance matrix

$$\mathbf{C}_{\mathbf{ee}} = \text{diag}([10^{-4} \quad 10^{-5} \quad 10^{-6} \quad 10^{-6} \quad 10^{-6}]). \quad (20)$$

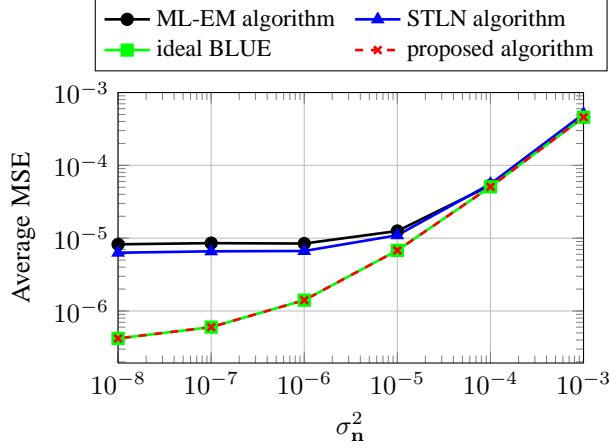


Fig. 1. Average MSEs of different iterative algorithms and the ideal BLUE in (18) plotted over the noise variance σ_n^2 . The model error variances have been kept constant.

For this model, the proposed algorithm in Sec. 3 was compared with the ideal BLUE in (18), the ML-EM algorithm and the STLN algorithm. For the latter one the l_2 norm minimization, a tolerance $\epsilon = 10^{-10}$ and $\mathbf{D} = \mathbf{I}^{5 \times 5}$ were chosen. Furthermore, \mathbf{X} (2.1) in [11] was identified to be the first N_h columns of $\mathbf{P}(\mathbf{x})$ in (13). For more details on these parameters we refer to [11]. For the ML-EM algorithm σ_n^2 was set to the mean value of \mathbf{V} [3]. The proposed algorithm and the ML-EM algorithm were executed for $N_{\text{iter}} = 10$ iterations. The resulting MSE values averaged over the elements of the MSE vector are presented in Fig. 1. This figure shows that the proposed algorithm attains the performance of the ideal BLUE and outperforms the competing algorithms especially for low σ_n^2 . The performance gain is more than one order of magnitude in MSE for small noise variances. For large noise variances, the ML-EM algorithm reaches the performance of the ideal BLUE as well. The reason for this is that the ML-EM algorithm incorporates the variances of both, the model errors and the noise. Hence, it recognizes that the model error variances are insignificant compared to those of the measurement noise. The STLN algorithm on the other hand does not incorporate the model error variances, decreasing its estimation accuracy. If one would have chosen \mathbf{C}_{ee} to be a scaled identity matrix, the STLN algorithm would have similar performance as the proposed algorithm for very low noise variances. The performance gain approximately stays the same for other values of \mathbf{x} .

For the next analysis, the noise variance was kept constant at $\sigma_n^2 = 10^{-6}$ and the accuracy of the estimated impulse response was varied by randomly choosing the diagonal elements of \mathbf{C}_{ee} from an uniform distribution between $[0, k]$. The parameter k on the other hand was varied between $k = 10^{-5}$ and $k = 10^{-2}$. The resulting MSE curves are plot-

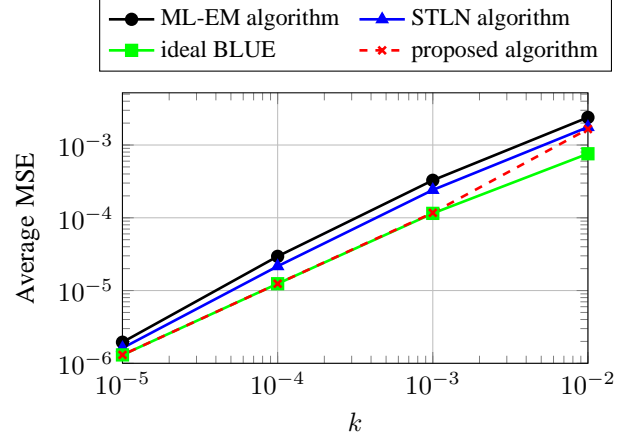


Fig. 2. Average MSEs of different iterative algorithms and the ideal BLUE in (18).

ted as a function of k in Fig. 2. Again, the proposed algorithm attains the performance of the ideal BLUE and outperforms the ML-EM and STLN algorithms for most values of k . For k smaller than 10^{-5} all algorithms perform approximately the same. For $k \geq 10^{-2}$ occasional divergence was observed for the proposed algorithm, leading to a decreased MSE performance. Again, the performance gain approximately stays the same for other values of \mathbf{x} .

5. CONCLUSIONS

In this work, a novel iterative algorithm for estimating an unknown but deterministic parameter vector in the presence of Gaussian model errors and Gaussian measurement noise is presented. This algorithm iteratively estimates the covariance matrix of an overall noise term, which describes the effects of the measurement noise as well as the noise resulting from the model uncertainty. This overall noise term was analyzed for unstructured model errors and for the case where the measurement matrix is a convolution matrix. For the latter case, simulation results are presented demonstrating the performance gain compared to the ML-EM algorithm and to the STLN algorithm achieved by the proposed algorithm for different noise and model error variances.

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